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On the non-uniqueness of minimal projection in L_p spaces

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Abstract

The main objective of this note is to exhibit a simple example of subspaces $U \subset L_p(\mu)$ $(p \neq 2)$ that admit two different projections with minimal norm. While for $p = 1, \infty$, such subspaces are well-known [W. Odyniec, G. Lewicki, Minimal Projections in Banach Spaces, in: Lecture Notes in Mathematics, vol. 1449, Springer-Verlag, Berlin, 1990. Problems of existence and uniqueness and their application], for $1 < p < \infty$ their existence was open.
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1. Introduction

Let U be a subspace of a Banach space X. A linear idempotent operator P on U is called a *projection* (onto U). The *relative projection constant* of U in X is defined as

$$\lambda(U, X) := \inf\{\|P\| : P \text{ is a projection of } X \text{ onto } U\}. \tag{1}$$

(If U is not complemented in X we set $\lambda(U, X) = \infty$). A projection P of X onto U is called a *minimal projection* if $\|P\| = \lambda(U, X)$. If U is a finite-dimensional or finite-codimensional space then a minimal projection of X onto U always exists [11]. In this note we are concerned with its uniqueness (see [4,7–9] for particular examples of unique minimal projections).

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In L_2 all minimal projections are unique. In L_1 and L_∞ the situation is drastically different. For any k, in both spaces there are a k-dimensional subspace and a k-codimensional subspace that admit two minimal projections. Cohen and Sullivan proved [5] that every subspace of L_p $(1 < p < \infty)$ that is the range of a contractive projection has a unique minimal projection. In particular, a minimal projection onto every one-dimensional subspace of L_p $(1 < p < \infty)$ is unique. This result was extended to two-dimensional subspaces in [16]. Odyniec [12] showed that every one-codimensional subspace of L_p $(1 < p < \infty)$ has a unique minimal projection.

In the case of ℓ_p^n $(1 < p < \infty, p \neq 2)$, combining the above theorems and Corollary 6 we obtain

Theorem 1. *Let* $1 < p < \infty, p \neq 2$.

(1) *For* $k = 1, 2, n-1$ *every* k*-dimensional subspace* U *of* ℓ_p^n *admits only one minimal projection.*
(2) *For* $k = 3, \dots, n-2$ *there is a* k*-dimensional subspace* U *of* ℓ_p^n *that admits two minimal projections.*

In the case of $L_p[a, b]$ $(1 < p < \infty, p \neq 2)$, combining the above theorems and Theorem 14 we obtain

Theorem 2. *Let* $1 < p < \infty, p \neq 2$.

(1) *For* $k = 1, 2$ *every* k*-dimensional subspace* U *of* $L_p[a, b]$ *admits only one minimal projection.*
(2) *For* $k \geq 3$ *there is a* k*-dimensional subspace* U *of* $L_p[a, b]$ *that admits two minimal projections.*
(3) *For* $k = 1$ *every* k*-codimensional subspace* U *of* $L_p[a, b]$ *admits only one minimal projection.*
(4) *For* $k \geq 3$ *there is a* k*-codimensional subspace* U *of* $L_p[a, b]$ *that admits two minimal projections.*

Even though we have found a two-codimensional subspace of ℓ_p^n that admits two minimal projections, we do not know the answer to this question in the case of two-codimensional subspaces of $L_p[a, b]$. $L_p[a, b]$ space is different from ℓ_p^n space because, in a non-atomic case, the space $L_p[a, b]$ is almost transitive (has “many” isometries). By contrast, ℓ_p^n is not (has only “few” isometries). We discuss this question in Section 3; in particular we prove that if $U = \ker\{f, g\}$ and $\operatorname{span}\{f, g\}$ has an isometry that is similar to a rotation, then U admits only one minimal projection (see Theorem 19). Our conjecture is that this is always the case.

2. Uniqueness in l_p^n

Let Y and Z be Banach spaces. For every p, $1 \leq p \leq \infty$, define a Banach space $Y \oplus_p Z$ as a vector space of all pairs (y, z), $y \in Y$, $z \in Z$, equipped with the norm

$$\|(y, z)\| := (\|y\| + \|z\|)^{\frac{1}{p}}. \tag{2}$$

Given two operators L and T on the spaces Y and Z respectively, we define an operator (L, T) on $Y \oplus_p Z$ by letting

$$(L, T)(y, z) = (Ly, Tz). \tag{3}$$

Lemma 3. *We have*

$$\|(L, T)\| = \max\{\|L\|, \|T\|\}. \tag{4}$$

Proof. First, observe that

$$\begin{aligned}\|(L, T)(y, z)\|^p &= \|Ly\|^p + \|Tz\|^p \le \|L\|^p \|y\|^p + \|T\|^p \|z\|^p \\ &\le (\max\{\|L\|, \|T\|\})^p (\|y\|^p + \|z\|^p) \\ &= (\max\{\|L\|, \|T\|\})^p \|(y, z)\|^p ,\end{aligned} \tag{5}$$

and hence $\|(L, T)\| \le \max\{\|L\|, \|T\|\}$.

Assume $\max\{\|L\|, \|T\|\} = \|T\|$ and let $z_n \in Z$ be such that $\|z_n\| = 1$ and $\|Tz_n\| \to \|T\|$. Then $\|(0, z_n)\| = 1$ and

$$\|(L, T)(0, z_n)\| \to \|T\| = \max\{\|L\|, \|T\|\} \tag{6}$$

which proves the lemma. ■

Lemma 4. *Let Y and Z be Banach spaces. Let V and W be any subspace of Y and Z respectively. Define*

$$U := V \oplus_p W := \{(v, w) : v \in V, w \in W\} \subset Y \oplus_p Z. \tag{7}$$

Then

$$\lambda(U, Y \oplus_p Z) = \max\{\lambda(V, Y), \lambda(W, Z)\}. \tag{8}$$

Additionally, if V is a proper *subspace of Y, and W is any subspace of Z such that*

$$\lambda(V, Y) < \lambda(W, Z), \tag{9}$$

and there exists a minimal projection R of Z onto W, then it follows that, for every $p \in [1, \infty]$, a minimal projection of $Y \oplus_p Z$ onto $U = V \oplus_p W$ is not unique.

Proof. Let Q and R be minimal projections onto V and W respectively. Form a projection (Q, R) of X onto U using

$$(Q, R)(y, z) := (Qy, Rz), \quad (y, z) \in X. \tag{10}$$

Then, by the previous lemma,

$$\|(Q, R)\| = \max\{\|Q\|, \|R\|\} = \|R\| = \lambda(W, Z). \tag{11}$$

We will now show that (Q, R) is a minimal projection of X onto U.

Let $J : X \to Z$ be a map defined by $J(y, z) = z$ and $E : Z \to X$ a map defined by $E(z) = (0, z)$. Clearly,

$$\|E\| = \|J\| = 1 \quad \text{and} \quad EJ = I. \tag{12}$$

Let S be another projection of X onto U. Since $JU \subset W$ it follows that $JSE : Z \to W$. Now for any $w \in W$ we have

$$JSEw = JS(0, w) = J(0, w)(\text{since } (0, w) \in U) = w \tag{13}$$

and JSE is a projection of Z onto W. Since R is a minimal projection onto W, we have

$$\|(Q, R)\| = \max\{\|Q\|, \|R\|\} = \|R\| \le \|JSE\| \le \|S\| \tag{14}$$

which shows that (Q, R) is a minimal projection onto U.

Now we will prove the second part of this lemma. Let Q be *any* projection (not necessarily minimal) of Y onto V with $\|Q\| \le \lambda(W, Z) = \|R\|$. The theorem follows from the fact that for two different projections Q_1 and Q_2 of Y onto V with

$$\|Q_i\| \le \lambda(W, Z), \quad i = 1, 2, \tag{15}$$

the corresponding projections (Q_1, R) and (Q_2, R) are also different. ■

Remark 5. The above result can also be proved using the notion of the Chalmers–Metcalf operators (see [3] and for further properties [10]). If R is a minimal projection of Z onto W then there is a Chalmers–Metcalf operator for R:

$$E = \int_{\mathcal{E}(R)} y \otimes x \, \mathrm{d}\mu : W \to W. \tag{16}$$

Then, by Lemma 3, if $\lambda(V, Y) \le \lambda(W, Z)$ and $\|Q\| \le \|R\|$ then

$$\widetilde{E} = \int_{\mathcal{E}(R)} (0, y) \otimes (0, x) \, \mathrm{d}\mu : V \oplus_p W \to V \oplus_p W, \tag{17}$$

will be a Chalmers–Metcalf operator for projection (Q, R) of $Y \oplus_p Z$ onto $V \oplus_p W$. As a result, any such projection (Q, R) will be minimal. One can easily show that if $\lambda(V, Y) < \lambda(W, Z)$ then all Chalmers–Metcalf operators are not invertible. This is related to [9] where it was shown that the invertibility of a Chalmers–Metcalf operator implies uniqueness. When $\lambda(V, Y) = \lambda(W, Z)$ then (see [9]) some of the Chalmers–Metcalf operators are invertible and some not, but in general the equation $\lambda(V, Y) = \lambda(W, Z)$ implies uniqueness of minimal projection.

As a corollary, we immediately get the following:

Corollary 6. *Let $1 \le p \le \infty$, $p \ne 2$. The following facts hold:*

(1) *There exists a three-dimensional subspace $U \subset l_p^5$ (and thus of codimension 2) such that a minimal projection onto U is not unique.*
(2) *For any $N \ge 5$ and any $3 \le n \le N-2$ there exists an n-dimensional subspace $U \subset l_p^N$ with a non-unique minimal projection.*
(3) *For any $N \ge 5$ and any $2 \le m \le N-3$ there exists a subspace $U \subset l_p^N$ of codimension m with a non-unique minimal projection.*
(4) *For any $n \ge 3$ and any $m \ge 2$ the infinite-dimensional space l_p contains subspaces U_n of dimension n and U_m of codimension m with non-unique minimal projections.*

Proof. (1) Since $l_p^5 = l_p^2 \oplus_p l_p^3$ by picking any one-dimensional subspace $V \subset l_p^2$, we are assured that $\lambda(V, l_p^2) = 1$. Let W be any two-dimensional subspace l_p^3 such that $\lambda(W, l_p^3) > 1$. For instance,

$$W := \{(b, c, d) \in l_p^3 : b + c + d = 0\}. \tag{18}$$

By [1], $\lambda(W, l_p^3) > 1$. As proved in [17],

$$\lambda(W, l_p^3) = \frac{1}{3}(1 + 2^{q/p})^{1/q}(1 + 2^{p/q})^{1/p}. \tag{19}$$

Then the Lemma 4 yields the result. A concrete example of one such space is

$$U = \{(0, a, b, c, d) \in l_p^5 : b + c + d = 0\}. \tag{20}$$

(2) Realizing l_p^N as

$$l_p^N = l_p^2 \oplus_p l_p^{N-2}, \tag{21}$$

it follows that $3 \leq N-2$. For $3 \leq n \leq N-2$ we pick an $(n-1)$-dimensional subspace $W \subset l_p^{N-2}$ and a one-dimensional subspace $V \subset l_p^2$ such that $\lambda(W, l_p^{N-2}) > 1$, (such a space exists; see [1] for $1 \leq p < \infty$ and [2] for $p = \infty$) and $\lambda(V, l_p^2) = 1$. Since V is a proper subspace of l_p^2, then Lemma 4 proves the statement.

The proofs for (3) and (4) are similar. ■

Combining Theorem 3.1 from [13] (which is a conglomeration of several theorems characterizing Hilbert spaces) with Lemma 4 we have the following:

Proposition 7. *Let Y be a Banach space with* $\dim Y \geq 2$ *and let Z be a Banach space not isomorphically isometric to a Hilbert space with* $\dim Z \geq 3$. *Let $p \in [1, \infty]$. Then $X := Y \oplus_p Z$ contains a subspace $U \subset Y \oplus_p Z$ with a non-unique minimal projection satisfying:*

(1) $\dim U = 3$*;*
(2) *(when* $\dim Z; \infty$ *and Z is strictly convex)* $\dim U = n$ *for any* $3 \leq n \leq \dim Z$*;*
(3) *(when* $\dim Y = 2$*) the codimension of U is two;*
(4) *(when* $\dim Y = 2, \dim Z; \infty$ *and Z is strictly convex)* $\operatorname{codim} U = n$ *for any* $2 \leq n \leq \dim Z - 1$

Proof. In each of these cases we choose a one-dimensional $V \subset Y$. Thus the above-mentioned theorem guarantees the existence of a subspace $W \subset Z$ of appropriate dimension or codimension with $\lambda(W, Z) > 1$. ■

3. Uniqueness in $L_p[a, b]$

In this section we will address the question of uniqueness of minimal projections in the non-atomic case. Since l_p imbeds isometrically into L_p as a range of a contractive projection, it follows that the results of the previous section concerning non-uniqueness of minimal projections onto finite-dimensional subspaces of l_p immediately extend to L_p. Hence for every $k > 2$ there exists a subspace $W \subset L_p$ with $\dim W = k$ that admits two minimal projections.

The case of finite codimension appears to be more complicated. The construction from the previous section used the existence of one-codimensional subspaces with different projection constants. This is not the case with L_p. As was shown in [15], all one-codimensional subspaces of L_p have the same projection constants. As a result, we make the following conjecture:

Conjecture 8. *Every subspace of $L_p[a, b]$ of codimension 2 has a unique minimal projection.*

At the end of this section we will present some corroborating evidence for the conjecture. The rest of this section focuses on proving non-uniqueness of minimal projections onto subspaces of codimension > 3. To accomplish this task in the spirit of the last section, we need to construct two subspaces of codimension 2 that have different projection constants. This task is not entirely trivial as Theorem II.8.4 [12] proves the existence of such subspaces only for p sufficiently close to 1 or ∞.

L_p space is different from ℓ_p^n space because, in a non-atomic case, the space $L_p[a, b]$ $(1 < p < \infty)$ has many isometries. In fact, in $L_p[a, b]$ $(1 < p < \infty)$ spaces the standard norm is *almost transitive* (see [14]). That is, for every $f \in S(L_p)$,

$$\operatorname{cl}\{I(f) : I \text{ is a linear isometry and onto}\} = S(L_p). \tag{22}$$

Almost transitivity implies asymptotic transitivity. That is for every $f \in S(L_p)$,

$$\bigcap_{\epsilon>0}\{T(f) : T \text{ is an isomorphism, } \max(\|T\|, \|T^{-1}\|) \leq 1+\epsilon\} = S(L_p). \tag{23}$$

These two properties imply (see Theorem 9) that every one-codimensional subspace of $L_p[a, b]$ has the same projection constant.

Let q be the conjugate to p: $\frac{1}{p} + \frac{1}{q} = 1$. For every non-zero functional $f \in L_q[a, b]$ we let

$$\ker f = \left\{x \in L_p[a, b] : \int_a^b f(t)x(t)\mathrm{d}t = 0\right\}. \tag{24}$$

Clearly, $\ker f$ is a one-codimensional subspace of $L_p[a, b]$ and every one-codimensional subspace of $L_p[a, b]$ is of this form. Any projection P onto $\ker f$ can be written as

$$P = Id - f \otimes x \tag{25}$$

with $f(x) = 1$. Using the same notation we let f stand for the function and the functional. For instance, if $f = 1$ we let

$$\ker(1) = \left\{x \in L_p[a, b] : \int_a^b x(t)\,\mathrm{d}t = 0\right\} \tag{26}$$

and for $f(t) = \cos t$

$$\ker(\cos t) = \left\{x \in L_p[a, b] : \int_a^b x(t)\cos t\,\mathrm{d}t = 0\right\}. \tag{27}$$

Theorem 9. *Let* $1 < p < \infty$ *and*

$$\alpha_p := \sup_{t\in(0,1)} (t^{p-1} + (1-t)^{p-1})^{1/p}(t^{q-1} + (1-t)^{q-1})^{1/q}. \tag{28}$$

Then:

(1) *(Franchetti [6]). The projection* $P = I - 1\otimes 1$ *is the minimal projection onto* $\ker 1 \subset L_p[0, 1]$ *and*

$$\|P\| = \lambda(\ker 1, L_p[0, 1]) = \alpha_p > 1. \tag{29}$$

(2) *(Rolewicz [15]) For every* $f \in L_q[a, b]$

$$\lambda(\ker f, L_p[0, 1]) = \alpha_p. \tag{30}$$

For every $f \in S(L_p)$ there is a unique norming functional $N_f \in S(L_q)$ such that

$$N_f(f) = 1, \quad (\text{or } f(N_f) = 1). \tag{31}$$

Next, we give the proof of the following observation from [18].

Lemma 10. *Let* $f \in S(L_p)$. *The unique minimal projection of* $L_p[0, 1]$ *onto* $\ker f$ *is*

$$P = Id - f \otimes N_f. \tag{32}$$

Proof. Let $1 < p < \infty$ and $Q = Id - 1 \otimes 1$. This is a minimal projection [15]. Using (23) there is a sequence of isomorphisms T_n such that

$$T_n(1) = N_f \quad \text{and} \quad \|T_n\| \cdot \|T_n{}^{-1}\| \to 1. \tag{33}$$

Let $f_n = 1 \circ T_n{}^{-1}$ and consider $P_n = Id - f_n \otimes N_f$. Since

$$T_n \circ Q \circ T_n{}^{-1} = Id - (1 \circ T_n{}^{-1}) \otimes T_n(1) = Id - f_n \otimes N_f(f) = P_n, \tag{34}$$

we have $\|P_n\| \to \|Q\|$. Additionally, since

$$f(N_f) = 1 \quad \text{and} \quad f_n(N_f) = 1 \text{ (observe that } \|f_n\| \to 1) \tag{35}$$

then $\|f_n - f\| \to 0$. As a result,

$$\|P_n\| = \|Id - f_n \otimes N_f\| \to \|Id - f \otimes N_f\| = \|P\|. \tag{36}$$

Therefore, $\|P\| = \|Q\| = \alpha_p$. As a result, P has minimal norm and thus is a minimal projection. ■

Lemma 11. *Let $1 < p < \infty$. The projection*

$$S = Id - \cos \pi t \otimes \cos \pi t \tag{37}$$

from $L_p[-1, 1]$ onto $\ker(\cos \pi t)$ is not minimal. Additionally, for $p \neq 2$,

$$\|S\| > \lambda(\ker(\cos \pi t), L_p[-1, 1]) \tag{38}$$

and there exists an even function w which is a norming point for S.

Proof. The fact that S is not minimal follows from Lemma 10. The projection P has a decomposition $S = (Q, R)$ on $L_p[-1, 1] = L_p[-1, 0] \oplus_p L_p[0, 1]$, where

$$\begin{aligned} &Q = Id - \cos \pi t \otimes \cos \pi t : L_p[-1, 0] \to L_p[-1, 0], \\ &R = Id - \cos \pi t \otimes \cos \pi t : L_p[0, 1] \to L_p[0, 1] \end{aligned} \tag{39}$$

and, by symmetry, $\|Q\| = \|R\|$. By Lemma 3,

$$\|S\| = \max\{\|Q\|, \|R\|\} = \|R\|. \tag{40}$$

Let z be a norming point for R. Then $z \in L_p[0, 1]$ and $\|Rz\| = \|R\|$. Define $y \in L_p[-1, 0]$ by $y(t) = z(-t)$. Once again, by symmetry z is a norming point for Q. Let

$$w = \frac{1}{2^{\frac{1}{p}}}(y, z). \tag{41}$$

We have $\|w\|^p = \frac{1}{2}(\|y\|^p + \|z\|^p) = 1$ and

$$\|Sw\|^p = \frac{1}{2}(\|Qy\|^p + \|Rz\|^p) = \|S\|^p \tag{42}$$

which proves the lemma. ■

The next lemma is an extension of Theorem II.8.4 [12] where the last assertion is made only for p sufficiently close to 1.

Lemma 12. *Let $1 < p < \infty$. The projection*

$$P = Id - \cos \pi t \otimes \cos \pi t - \sin \pi t \otimes \sin \pi t \tag{43}$$

is the minimal projection of $L_p[-1, 1]$ onto the space

$$U := \ker(\cos \pi t) \cap \ker(\sin \pi t). \tag{44}$$

The set of all norming points for P contains an even function and for $p \neq 2$, the norm $\|P\| > \alpha_p$.

Proof. The minimality of P follows the standard averaging argument. Let z be a norming point for P. Since $\|P\| > 1$, it follows that at least one of the values $\int_{-1}^{1} x(t) \cos \pi t \, dt$ or $\int_{-1}^{1} x(t) \sin \pi t \, dt$ is different from zero. Hence, there exist linearly independent functions

$$\begin{aligned} f_1(t) &= b \cos \pi t - b \sin \pi t = \cos \pi(t + \theta) \\ f_2(t) &= a \cos \pi t + b \sin \pi t = \sin \pi(t + \theta) \end{aligned} \tag{45}$$

such that $\int_{-1}^{1} f_2(t) z(t) dt = 0$. Since $\ker f_1 \cap \ker f_2 = U$, it follows that

$$P = Id - \cos \pi(t + \theta) \otimes \cos \pi(t + \theta) - \sin \pi(t + \theta) \otimes \sin \pi(t + \theta) \tag{46}$$

and

$$\int_{-1}^{1} z(t) \sin \pi(t + \theta) dt = 0. \tag{47}$$

By translation invariance, the function $z_1(t) := z(t - \theta)$ is a norming point for P and

$$\int_{-1}^{1} z_1(t) \sin \pi t \, dt = \int_{-1}^{1} z(t) \sin \pi(t + \theta) dt = 0. \tag{48}$$

Thus,

$$\|P\| = \|P z_1\| = \left\| z_1 - \left(\int_{-1}^{1} z_1(t) \cos \pi t \, dt \right) \cos \pi t \right\| \leq \|S\| \tag{49}$$

where S is defined by (37). By Lemma 11, there exists an even function w that is a norming point for S. From $\int_{-1}^{1} w(t) \sin \pi t \, dt = 0$ we deduce $Pw = Sw$ and

$$\|S\| = \|Sw\| = \|Pw\| \leq \|P\|; \tag{50}$$

which, together with (49), shows that w is also a norming point for P. The inequality (38) completes the proof of the lemma. ■

Corollary 13. *Theorem 9 cannot be extended to subspaces of $L_p[-1, 1]$ of codimension 2. That is, there exist two different subspaces U and W in $L_p[-1, 1]$ ($1 < p < \infty$, $p \neq 2$) such that*

$$\operatorname{codim} U = \operatorname{codim} W = 2 \tag{51}$$

and

$$\lambda(U, L_p[-1, 1]) > \lambda(W, L_p[-1, 1]) = \alpha_p. \tag{52}$$

Proof. Make the decomposition

$$L_p[-1, 1] = L_p[-1, 0] \oplus_p L_p[0, 1] \tag{53}$$

and let $W_1 \subset L_p[-1,0]$ and $W_2 \subset L_p[0,1]$ be one-codimensional subspaces, and let $W = W_1 \oplus_p W_2 \subset L_p[-1,1]$. Then the codimension of W is 2. Combining Lemma 4 and Theorem 9 we have

$$\lambda(W, L_p[-1,1]) = \max\{\lambda(W_1, L_p[-1,1]), \lambda(W_2, L_p[-1,1])\} = \alpha_p. \tag{54}$$

For instance, $W = \ker \chi_{[-1,0]} \cap \ker \chi_{[0,1]}$ is sufficient. Thus, it is enough to choose U as in the previous lemma. ■

As another corollary, we obtain the following non-atomic version of the Corollary 6.

Theorem 14. *Let $1 \le p \le \infty$, $p \ne 2$*

(1) *For every $m \ge 3$, there exists a subspace $U \subset L_p[-1,1]$ with $codim\, U = m$ such that the minimal projection onto U is not unique.*
(2) *For every $n \ge 3$ there exists a subspace $U \subset L_p[-1,1]$ with $\dim U = n$ such that the minimal projection onto U is not unique.*

Proof. (1) Make the decomposition

$$L_p[-1,1] = L_p[-1,0] \oplus_p L_p[0,1]. \tag{55}$$

Choose any $V \subset L_p[-1,0]$ with $codim\, V = m-2$. By Corollary 13, we can choose a two-codimensional subspace $W \subset L_p[0,1]$ such that $\lambda(W, L_p[0,1]) \ne \lambda(V, L_p[-1,0])$. By Lemma 4, the space $U = V \oplus_p W$ is the desired subspace.

(2) Choose a one-dimensional subspace $V \subset L_p[-1,0]$ and a subspace W in $L_p[0,1]$ with $\dim W = n-1$ and $\lambda(W, L_p[0,1]) > 1$. ■

The problem of uniqueness of minimal projections onto two-codimensional subspaces of L_p is open. Some partial results are as follows.

To every minimal projection there corresponds a Chalmers–Metcalf operator (see [3] and for further discussion [10]). As a result [10], if the minimal projection is not unique then all minimal projections have many norming pairs in common.

Theorem 15. *Let $1 < p < \infty$ and let V be a two-codimensional subspace in L_p. If P and Q are two minimal projections of L_p to V then they can be written as*

$$\begin{aligned} P &= Id - f \otimes y - g \otimes z_1 \\ Q &= Id - f \otimes y - g \otimes z_2, \end{aligned} \tag{56}$$

where $V = \ker(f) \cap \ker(g)$ and $f(y) = 1, g(y) = 0, f(z_i) = 0, g(z_i) = 1$.

Additionally,

$$g(x_0) = 0 \tag{57}$$

for any x_0 that is in a common norming pair for P and Q.

Proof. Let $P = Id - f \otimes y_1 - g \otimes x_1$ and $Q = Id - f \otimes y_2 - g \otimes x_2$. By [10], P and Q must have at least one norming pair in common. That is,

$$h(P(x_0)) = h(Q(x_0)) = \|P\| = \|Q\|, \quad \text{for some } h \in S(V^*) \text{ and } x_0 \in S(L_p). \tag{58}$$

Since L_p is a strictly convex space, it follows that

$$P(x_0) = Q(x_0), \quad \text{for some } x_0 \in S(L_p). \tag{59}$$

That implies

$$x = f(x_0)y_1 + g(x_0)x_1 = f(x_0)y_2 + g(x_0)x_2. \tag{60}$$

$(f(x_0))^2 + (g(x_0))^2 \neq 0$; otherwise the norm of P and Q would have to be equal to 1. However this would imply that $P = Q$ (as we have uniqueness of norm 1 projections in smooth spaces [5]). Take

$$z_1 = g(x_0)y_1 - f(x_0)x_1, \qquad z_2 = g(x_0)y_2 - f(x_0)x_2. \tag{61}$$

It is easy to see that $\ker P = \operatorname{span}\{y_1, x_1\} = \operatorname{span}\{x, z_1\}$, and $\ker Q = \operatorname{span}\{y_2, x_2\} = \operatorname{span}\{x, z_2\}$. Let

$$\alpha = \frac{f(x_0)}{(f(x_0))^2 + (g(x_0))^2}, \qquad \beta = \frac{g(x_0)}{(f(x_0))^2 + (g(x_0))^2} \tag{62}$$

and

$$h = \alpha f + \beta g, \qquad k = \beta f - \alpha g. \tag{63}$$

One can easily determine that

$$h(x) = 1, \quad h(z_i) = 0, \quad k(x) = 0, \quad k(z_i) = 1. \tag{64}$$

Therefore,

$$\begin{aligned} P &= Id - h \otimes x - k \otimes z_1 \\ Q &= Id - h \otimes x - k \otimes z_2, \end{aligned} \tag{65}$$

which proves the first part. The second part follows from the fact that $P(x_0) = Q(x_0)$ implies $k(x_0)z_1 = k(x_0)z_2$. Thus, either $z_1 = z_2$ (and then $P = Q$) or $k(x_0) = 0$. ■

The proof that we have uniqueness of one-codimensional minimal projections in strictly convex spaces is almost the same as that for the above theorem, and therefore we will state it below, for the sake of completeness.

Theorem 16 (*Odyniec [12]*). *Let V be a one-codimensional subspace of a strictly convex space X. If $\lambda(V, X) > 1$ then a minimal projection of X to V is unique.*

Proof. Let $V = \ker f$. If we have two minimal projections, for instance $P = Id - f \otimes y_1$ and $Q = Id - f \otimes y_2$, then [10] they must have at least one norming pair in common. That is

$$h(P(x_0)) = h(Q(x_0)) = \|P\| = \|Q\|, \quad \text{for some } h \in S(V^*) \text{ and } x_0 \in S(L_p). \tag{66}$$

Strict convexity of X implies $P(x_0) = Q(x_0)$. Since $\lambda(V, X) > 1$ then $x_0 \notin V$. As a result $P = Q$. ■

Remark 17. The above theorem is false if $\lambda(V, X) = 1$. For example, let

$$S(X) = \{(x, y) : y^2 = (x^2 - 1)^2 \text{ and } -1 \le x \le 1\} \tag{67}$$

and

$$V = \{(x, y) : y = 0\}. \tag{68}$$

Then $P(x, y) = (x, 0)$ and $Q(x, y) = (x - \frac{y}{2}, 0)$ both have norm equal to 1. Thus P and Q are both minimal. The above situation cannot happen if X is a smooth space (for instance L_p is both a smooth and a strictly convex space) as by [5] all minimal projections of norm 1 in smooth space X are unique.

Lemma 18. *Let $T^* : L_q \to L_q$ be an onto isometry such that*

$$T^*(\mathrm{span}\{f, g\}) = \mathrm{span}\{f, g\}. \tag{69}$$

Then $T = T^{**} : L_p \to L_p$ is an onto isometry such that

$$T(\ker f \cap \ker g) = \ker f \cap \ker g. \tag{70}$$

Proof. If T^* is an isometry then $T = T^{**}$ is also an isometry. To prove the second part, take x such that $x \in \ker f \cap \ker g$. Then, because of (69), $f(Tx) = T^* f(x) = 0$ and $g(Tx) = T^* g(x) = 0$. That is, $Tx \in \ker f \cap \ker g$. What is more, take $x \ker f \cap \ker g$ and x_0 such that $T(x_0) = x$. We have $0 = f(x) = f(Tx_0) = T^* f(x_0)$ and $0 = g(x) = g(Tx_0) = T^* g(x_0)$. Once again, because of (69), this implies $x_0 \in \ker f \cap \ker g$. ■

Theorem 19. *Let $1 < p < \infty$ and let $V = \ker f \cap \ker g$. Assume that $T^* : L_q \to L_q$ is an onto isometry such that (69) holds and $T^*|_{\mathrm{span}\{f,g\}}$ does not have any non-zero eigenvector. Then the minimal projection of L_p onto V is unique.*

Proof. Assume that R and S are two distinct minimal projections of L_p onto V. Then $T^{-1} \circ R \circ T$ and $T^{-1} \circ S \circ T$ are also projections of L_p onto V. Therefore,

$$Q = \frac{R + T^{-1} \circ R \circ T + S + T^{-1} \circ S \circ T}{4} \tag{71}$$

is also a minimal projection of L_p onto V.

Let x be a norming point for Q, that is $\|Q(x)\| = \|P\|$. Then $\|R(x)\| = \|R\|$, $\|R(Tx)\| = \|R\|$, $\|S(x)\| = \|S\|$ and $\|S(Tx)\| = \|S\|$. As a result there is an $x \in L_p$ such that x and Tx are both norming points for R and S.

By Theorem 15, there are $k, l \in L_q$ such that $\mathrm{span}\{k, l\} = \mathrm{span}\{f, g\} = V$ and R, S can be written as in (56). Additionally, by (57)

$$k(x) = 0 \quad \text{and} \quad k(Tx) = 0. \tag{72}$$

By (69), $T^*k = \alpha k + \beta l$ and, since T^* does not have a non-zero eigenvector, $\beta \neq 0$. Then

$$0 = k(Tx) = T^*k(x) = \alpha k(x) + \beta l(x) = \beta l(x), \tag{73}$$

which would yield $l(x) = 0$. This can only be possible if the norm of minimal projection onto V is 1. Then by [5], the minimal projection onto V would be unique. ■

Corollary 20. *Let $1 < p < \infty$. The projection*

$$P = Id - \cos \pi t \otimes \cos \pi t - \sin \pi t \otimes \sin \pi t \tag{74}$$

is the unique minimal projection of $L_p[-1, 1]$ onto the space $U = \ker(\cos \pi t) \cap \ker(\sin \pi t)$.

Proof. Consider the isometry T^* given by the formula

$$T^*(f)(t) = f\left(t + \frac{1}{2}\right) \quad \text{modulo } [-1, 1]. \tag{75}$$

It is easy to check that it satisfies all the conditions from Theorem 19. ■

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